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Maximal Indecomposable Past Sets and Event Horizons

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The existence of maximal indecomposable past sets MIPs is demonstrated using the Kuratowski–Zorn lemma. A criterion for the existence of an absolute event horizon in space-time is given in terms of MIPs and a relation to black hole event horizon is shown.

1. INTRODUCTION

Geroch, Kronheimer, and Penrose (1972) have shown, under very general assumptions, that we can assign a boundary to space-time using certain past and future sets. In this paper we exploit the natural ordering induced by the causality relation on those past sets. Application of the Kuratowski–Zorn lemma gives us the existence of certain past sets the presence of which allows us to ascertain the existence of regions of space-time that will never be seen by an observer no matter how he chooses to move.

2. THE EXISTENCE OF MIPS

In the following by space-time $(\mathfrak{M}, \bar{g})$ we shall mean a Hausdorff, connected, C^2 manifold $\mathfrak{M}$ without boundary which admits a Lorentz metric $\bar{g}$. We shall assume that $(\mathfrak{M}, \bar{g})$ is past distinguishing, i.e., such that for points $p, q \in \mathfrak{M}$, $p \neq q$ implies $I^-(p) \neq I^-(q)$. A subset P of $\mathfrak{M}$ is a past set if it coincides with its own chronological past $I^-(P)$ and it is indecomposable if it is not empty and it cannot be expressed as a union of two proper subsets which are themselves past sets (Geroch et al., 1972). Following Geroch et al. (1972) we shall call such sets IPs.

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Lemma 1. Let $\tilde{\mathcal{M}}$ be the set of all IPs of $\mathcal{M}$; then $\tilde{\mathcal{M}}$ is partially ordered by inclusion and inductive.

Proof. Since for any $P, Q, R \in \tilde{\mathcal{M}}$ we have (1) $P \subset P$; (2) if $P \subset Q$ and $Q \subset P$ then $P = Q$; (3) if $P \subset Q$ and $Q \subset R$ then $P \subset R$; $\tilde{\mathcal{M}}$ is partially ordered by inclusion.

Consider any chain in $\tilde{\mathcal{M}}$, that is a subset $\tilde{\mathcal{M}}_o$ of $\tilde{\mathcal{M}}$ such that for any two elements X, Y of $\tilde{\mathcal{M}}_o$ either $X \subset Y$ or $Y \subset X$. Take the union $\tilde{U}_o$ of all elements of the chain $\tilde{\mathcal{M}}_o$. Clearly $\tilde{U}_o$ is a past set. It must also be indecomposable otherwise there exist past sets Q and R such that $\tilde{U}_o = Q \cup R$ and we can find points $q, r \in \mathcal{M}$ such that $q \in Q - R$ and $r \in R - Q$. However, we must have $q \in X$ and $r \in Y$ where X and Y are IPs of the chain $\tilde{\mathcal{M}}_o$. But either $X \subset Y$ or $Y \subset X$, therefore either $r \in I^-(q)$ or $q \in I^-(r)$, and consequently either Q or R contains both q and r, but this contradicts the construction of q and r. Thus we have shown that $\tilde{\mathcal{M}}$ is a partially ordered set such that for every chain $\tilde{\mathcal{M}}_o$ of $\tilde{\mathcal{M}}$ there exists an element X_o of $\tilde{\mathcal{M}}$ such that for any $X \subset \tilde{\mathcal{M}}_o$, $X \subset X_o$. By definition (Maurin, 1976, Chapter 1, p. 18) it follows that $\tilde{\mathcal{M}}$ is inductive. ■

An element M of $\tilde{\mathcal{M}}$ such that M is not a proper subset of any element of $\tilde{\mathcal{M}}$ is said to be maximal. By Lemma 1 and the Kuratowski–Zorn lemma (Maurin, 1976, Chapter 1, p. 18) we immediately have the following:

Theorem 1. Let $\tilde{\mathcal{M}}$ be the set of all IPs in $\mathcal{M}$ then there exists a maximal element in $\tilde{\mathcal{M}}$.

We shall call the maximal indecomposable past sets MIPs.

The IPs can be divided into two classes, consisting of those which are of the form $I^-(p)$ for some $p \in \mathcal{M}$, called proper IPs or PIPs, and those which are not, called terminal IPs or TIPs (Geroch et al., 1972).

Lemma 2. Let M be a MIP in $\tilde{\mathcal{M}}$ then it must be a TIP.

Proof. Suppose that M is a PIP; then $M = I^-(p)$, the chronological past of some point $p \in \mathcal{M}$ [Geroch et al., 1972, Theorem (2.3)]. Thus there exists a neighborhood $\mathcal{U}$ of p in $\mathcal{M}$ and consequently a point q in $\mathcal{U}$ to the future of p such that M is a proper subset of $I^-(q)$ since $\mathcal{M}$ is past distinguishing. This contradicts maximality of M. ■

3. MIPs AND ABSOLUTE EVENT HORIZONS

We know that TIPs define the future causal boundary (c boundary) of space-time (Geroch et al., 1972). We can think of MIPs as forming the essential part of the c boundary in the sense that any point of $\mathcal{M}$ must be in some MIP:

> *Theorem 2.* A point p belongs to some MIP of $\tilde{\mathfrak{M}} \Leftrightarrow p$ belongs to the space-time manifold $\mathfrak{M}$.

Proof. $\Rightarrow$ If p belongs to some MIP of $\tilde{\mathfrak{M}}$ it obviously must belong to $\mathfrak{M}$ since MIPs are subsets of $\mathfrak{M}$.

$\Leftarrow$ Any point of $\mathfrak{M}$ is in some IP of $\mathfrak{M}$. Therefore it is enough to prove that any IP is a subset of some MIP. Suppose that there is a nonempty subset $\tilde{\mathfrak{N}}$ of IPs of $\tilde{\mathfrak{M}}$ that do not belong to any MIP. The set $\tilde{\mathfrak{N}}$ forms again a set partially ordered by inclusion. $\tilde{\mathfrak{N}}$ must also be inductive otherwise there exists an IP not in $\tilde{\mathfrak{N}}$ that contains some chain $\tilde{\mathfrak{N}}_o$ from $\tilde{\mathfrak{N}}$, but since any IP not in $\tilde{\mathfrak{N}}$ is a subset of some MIP so is $\tilde{\mathfrak{N}}_o$ which contradicts the construction of $\tilde{\mathfrak{N}}$. Thus by the Kuratowski–Zorn lemma there exists a maximal element M' in $\tilde{\mathfrak{N}}$. This IP is not a proper subset of any IP in $\tilde{\mathfrak{N}}$ nor of any IP in $\tilde{\mathfrak{M}} - \tilde{\mathfrak{N}}$ thus by definition $\tilde{M}$ is a MIP in $\tilde{\mathfrak{M}}$. This contradicts the existence of the non-empty set $\tilde{\mathfrak{N}}$. ∎

Any TIP is the chronological past of some future-endless timelike curve [Geroch et al., 1972, Theorem (2.1)]. The boundary of a TIP is called the event horizon since it defines limits of regions of space-time that will never be visible by observers moving along causal curves (see Geroch et al., 1972, and Hawking and Ellis, 1973, p. 129). Some space-times possess the property that no matter how an observer O chooses to move there always be regions of space-time that will never be observable by O. In such a case we can say that there exists an absolute event horizon. The absolute event horizon can be thought of as giving event horizons for timelike geodesics (see Hawking and Ellis, p. 129). The main result of this paper is that MIPs can be used to define absolute event horizons.

Definition 1. The boundary in $\mathfrak{M}$ of a TIP is an absolute event horizon if and only if it is a MIP.

Intuitively we associate the existence of an absolute event horizon with the presence of spacelike components in the boundary of space-time (Hawking and Ellis, 1973, p. 130). MIPs can be used to make the concept of space-like point of the c boundary precise:

Definition 2. A TIP defines a spacelike c boundary point if and only if it is a MIP.

We have the following criterion for the existence of a nonempty absolute event horizon.

> *Theorem 3* (cf. Seifert, 1971, Theorem 6.2). There exists a nonempty absolute event horizon in $(\mathfrak{M}, \bar{g}) \Leftrightarrow$ there exist two MIPs in $\tilde{\mathfrak{M}}$.

Proof. $\Rightarrow$ Suppose that there exists only one MIP in $\bar{\mathcal{M}}$ then from Theorem 2 it must coincide with the space-time manifold $\mathcal{M}$ and thus its boundary in $\mathcal{M}$ must be empty.

$\Leftarrow$ If there were two MIPs in $\bar{\mathcal{M}}$ then by maximality neither is all of $\mathcal{M}$. So neither can have empty boundary in $\mathcal{M}$. ■

The MIPs of the Schwarzschild space-time (Hawking and Ellis, 1973, Figure 24, p. 154) consists of pasts of all points of the future singularity $r = 0$ and also of pasts of two points denoted by i^+ which are end points of future-endless timelike curves.

A general class of space-times that do not admit any absolute event horizon are asymptotically simple and empty (ASE) space-times (for definition see Hawking and Ellis, 1973, p. 222). This follows from the argument of Hawking and Ellis (1973) on pp. 224 and 225 where the existence is demonstrated of a TIP that coincides with the whole space-time manifold.

Any null geodesic in an ASE space-time has a future end point on part of its boundary $\partial\mathcal{M}$ denoted by $\mathcal{I}^+$ and called future null infinity. It is clear that then $J^-(\mathcal{I}^+, \mathcal{M})$, the causal past of $\mathcal{I}^+$ in $\mathcal{M}$, must coincide with $\mathcal{M}$ itself.

4. MIPs AND BLACK HOLE EVENT HORIZONS

One would like to know whether there is a relationship between an absolute event horizon and a black hole event horizon. The latter is defined in a weakly asymptotically simple and empty space-time as the boundary $\dot{J}^-(\mathcal{I}^+, \mathcal{M})$. A space-time $(\mathcal{M}, g)$ is called weakly asymptotically simple and empty (WASE) if there is an ASE space-time $(\mathcal{M}', g,)$ and a neighborhood $\mathcal{U}'$ of $\partial\mathcal{M}'$ in $\mathcal{M}'$ such that $\mathcal{U}' \cap \mathcal{M}'$ is isometric to an open set $\mathcal{U}$ of $\mathcal{M}$ (Hawking and Ellis, 1973, p. 225). Indeed, in the Schwarzschild space-time, $\dot{J}^-(\mathcal{I}^+, \mathcal{M})$ is the boundary of one of two MIPs. These MIPs are pasts of two points i^+ and they are unique MIPs such that the null generators of their boundaries are not converging. However, it is easy to convince oneself that $I^-(\mathcal{I}^+, \mathcal{M})$ is no longer a MIP if we modify the Schwarzschild space-time so that its boundary at the singularity becomes null. Another case is the maximally extended Reissner–Nordström solution (Hawking and Ellis, 1973, Figure 25, p. 158) where the MIP coincides with the whole of the space-time manifold. Nevertheless we can prove the following:

> *Theorem 4.* Let $\dot{J}^-(\mathcal{I}^+, \mathcal{M})$ be the black hole event horizon in a WASE space-time $(\mathcal{M}, g)$; then $I^-(\mathcal{I}^+, \mathcal{M})$, the chronological past of $\mathcal{I}^+$ in $\mathcal{M}$, is a TIP in $\bar{\mathcal{M}}$ and there exists a timelike curve λ such that $I^-(\lambda) = I^-(\mathcal{I}^+, \mathcal{M})$ and an isometry map f from an open set in

$\mathcal{M}$ to an open neighborhood of $\partial\tilde{\mathcal{M}}'$ in an ASE space-time $(\mathcal{M}', \bar{g}')$ such that $I^-(f[\lambda])$ is a MIP in $\tilde{\mathcal{M}}'$.

Proof. By definition of the WASE space-time there exists a corresponding space-time $(\mathcal{M}', \bar{g}')$. The MIP of $(\mathcal{M}', \bar{g}')$ coincides with $\mathcal{M}'$ itself and by Theorem (2.1) of Geroch et al. (1972) it is of the form $I^-(\lambda'')$ for some future-endless timelike curve λ'' in $\mathcal{M}'$. Consider the intersection λ' of λ'' with the neighborhood $\mathcal{U}$ of $\partial\mathcal{M}'$ in the definition of the WASE space-time, we still have $I^-(\lambda') = \mathcal{M}'$. Since $\mathcal{U}' \cap \mathcal{M}'$ is isometric to an open set $\mathcal{U}$ in $\mathcal{M}$ we have that $\lambda' = f[\lambda]$ where f is the isometry map and λ is a future-endless timelike curve in $\mathcal{M}$. $I^-(\lambda)$ is a TIP in $\mathcal{M}$ and it must coincide with $I^-(\mathcal{J}^+, \mathcal{M})$ by definition of the WASE space-time. ∎

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